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On Diffusion Equations for Dynamical Systems Driven by Noise

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A unified formalism is presented to study Hamiltonian linear systems driven by noise. With this formalism, the phase averaging approximation, valid at weak noise, is easily performed. Already known results are straightforwardly recovered and new ones are obtained. After introducing this formalism on the exactly solvable one-degree-of-freedom problem with uncorrelated noise, one studies the corresponding exponentially correlated case. The validity of the approximate results thus obtained is considered by investigating the systematic weak-disorder expansion beyond the quasilinear approximation. In particular, it is argued that this expansion behaves uniformly for weak and large correlation time. The two-degrees-of-freedom problem is completely solved at the low-disorder approximation and this result is applied to the two-channel Anderson localization problem. The invariant measure and the two positive Lyapunov exponents are computed at all coupling between the channels. For systems with n degree of freedom the phase averaging leads to a Fokker–Planck equation for the measure in action space describing the system. However, it is argued that it is not solvable except in a special case which is explicitly displayed and solved. Nevertheless, in the large-n limit, it is possible to compute the largest Lyapunov exponent. Moreover, generalized Lyapunov exponents are calculated in this limit, and they do not exhibit a dispersion: in particular, $\log\langle\mathscr{E}\rangle/\langle\log\mathscr{E}\rangle \sim 1$, where $\mathscr{E}$ is the energy of the system and where the brackets denote averaging over the noise. On the other hand, it is possible to compute at weak noise the sum of all the positive Lyapunov exponents. Taking into account all these results allows more insight on the whole spectrum of Lyapunov exponents.

KEY WORDS: Fokker–Planck equations; Lyapunov exponents; localization; parametric oscillators.

1. INTRODUCTION

Physical systems driven by noise constitute a common field of interest. More specifically, one can consider a linear dynamical system described by

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a time-independent unperturbed Hamiltonian H_0 to which one adds a stochastic perturbation $V(t)$. This perturbation may have time correlations. Such a model describes, for instance, a system of coupled harmonic oscillators driven parametrically by a noise. It describes also another problem of great interest, namely the Anderson localization in a system of mutually coupled continuous chains.[1,2] In this last case, the spatial coordinate along the chains plays the role of time.

The large-time behavior of this kind of system is characterized by the set of Lyapunov exponents.[2] In the case of coupled oscillators, the largest of these exponents determines the growth rate of the system's energy. For the localization problem, the smallest positive Lyapunov exponent is the inverse of the localization length, while the conductivity of a finite sample is related to the whole set of Lyapunov exponents. The properties of the Lyapunov spectrum have been recently studied by many authors, most of these works being numerical.[3,4] Exact results on this spectrum are rare and have been obtained in special cases (for example, see ref. 5).

In the white noise case, the mathematical methods used to study such systems amount to writing a Fokker–Planck equation for a measure on relevant quantities such as the transfer matrix in the localization problem. Unfortunately, only the one-dimensional case is exactly solvable.[5] However, interesting results have been obtained in the weak noise limit, for example, by Dorokhov[1] and Douçot and Rammal.[6]

All these methods ultimately lead to the determination of an invariant measure on a sphere in the phase space of the system by solving a second-order partial differential equation on this sphere. In a second step, this invariant measure allows one to compute the Lyapunov exponent.[8]

The aim of this paper is to present these techniques in the weak noise limit, using a unified formalism. This allows one to recover straightforwardly some already known results but also to derive new ones. For weak noise, the usual approximation amounts to perform phase averaging. Thanks to the Hamiltonian form of the system under consideration, one works in action–angle variables. This leads, in a systematic way, to diffusion equations in the action variables, which are algebraically simple. These equations can be set in a form which is general and independent of the particular statistic of the noise (Gaussian or correlated). Using the terminology of plasma physics, we call this phase averaging method the quasilinear approximation.

This paper is organized as follows. Section 2 is devoted to the one-dimensional case. The well-known exact result of Halperin[6] is recovered under the action–angle parametrization. Although this uncorrelated case is exactly solvable at all disorder, we present its quasilinear approximation in view of its pedagogical interest.

The one-dimensional correlated case is considered in Section 2.2. This problem is not exactly solvable. However, thanks to its structure, a perturbative treatment may be performed and this allows one to control the quasilinear approximation. Our conclusion is that for a given noise amplitude, the quasilinear approximation is uniform in the correlation time.

In Section 3 the quasilinear method is derived for multidimensional systems. The quasilinear diffusion equation is written for a noise with a general structure. For linear systems, one obtains a homogeneous diffusion equation which provides, in principle, a way to compute the invariant measure to which the largest Lyapunov exponent is related, on a sphere in action space.

For the sake of completeness, we recall in Section 3.3 the computation of the Kolmogorov entropy, a result which has been already published elsewhere.[9] It is shown that the Kolmogorov entropy is simply related to the coefficients entering the quasilinear diffusion equation.

All these results are then applied to the computation of the two positive Lyapunov exponents of the two-dimensional case, recovering and generalizing in a very simple way a result of Dorokhov.[10]

For n degrees of freedom ($n > 2$) it is argued in Section 4.1 that the problem in general is not solvable in general, even in the quasilinear limit, except in an isolated case which is solved in Section 4.2 within this limit. This special case presents interesting properties. Indeed, we find an absence of dispersion in the large-n limit, that is, for a physical quantity X (such as the energy of the system of oscillators) $\log\langle X\rangle/\langle\log X\rangle \sim 1$.

Finally, we consider the thermodynamic limit where the number of degrees of freedom goes to infinity. In the cases where no restriction arises from the quasidegeneracy of the eigenfrequencies this limit still allows a quasilinear approach. It is shown that the absence of dispersion is then a general feature. It is indeed the situation in the localization problem, as previously presented in ref. 9.

2. SYSTEMS WITH ONE DEGREE OF FREEDOM

2.1. One Degree of Freedom with Uncorrelated Gaussian Noise

In this part we will introduce the formalism on the simple example of a Hamiltonian system with one degree of freedom perturbed by a white noise potential. The aim of this part is essentially pedagogical. Well-known results in localization theory[9] will be recovered easily.

Let us consider a random frequency oscillator whose Hamiltonian reads

$$H(x, p, t) = \tfrac{1}{2}p^2 + \tfrac{1}{2}Ex^2 - \tfrac{1}{2}x^2V(t) \tag{1}$$

where $E > 0$ and

$$\langle V(t)\, V(t') \rangle = 2D\delta(t - t') \tag{2}$$

The action–angle variables (I, ϕ) of the system are given by

$$E^{-1/4}p = (2I)^{1/2} \sin \phi, \qquad E^{1/4}x = (2I)^{1/2} \cos \phi \tag{3}$$

In these variables the equations of motion read

$$\dot{I} = 2I \frac{V}{E^{1/2}} \sin \phi \cos \phi, \qquad \dot{\phi} = E^{1/2}\left(1 + \frac{V}{E} \cos^2 \phi\right) \tag{4}$$

Let us denote by $P(I, \phi, t)$ the measure on its phase space. Then, the Lyapunov exponent $\Lambda(D, E)$ is defined in term of $P(I, \phi, t)$ by

$$\Lambda(D, E) = \frac{1}{2} \lim_{t \to \infty} \partial_t \int P(I, \phi, t) \log I \, dI \, d\phi \tag{5}$$

It is easy to see that $\Lambda(D, E)$ scales as

$$\Lambda(D, E) = E^{1/2} \Lambda(D/E^{3/2}, 1) \tag{6}$$

So, in the following one sets $E = 1$ (except of course in the special case $E = 0$; see below).

One looks for a partial differential equation satisfied by the measure $P(I, \phi, t)$. For the more general stochastic equation $\dot{X}_i = f_i(X) + g_i^\alpha(X)\, V_\alpha(t)$ where $V_\alpha(t)$ are independent white noises, the measure satisfies the Fokker–Planck equation:

$$\partial_t P + \partial_i(f_i P) = \sum_\alpha D_\alpha(\partial_i g_i^\alpha)(\partial_j g_j^\alpha) P$$

(here the stochastic equation is interpreted in the Stratanovich sense). In particular, (4) leads to the Fokker–Planck equation

$$\partial_t P + \partial_\phi P = D(2c^2 I \partial_I P + 4I^2c^2s^2\partial_I^2 P + 4Isc^3\partial_I\partial_\phi P - 2c^3s\partial_\phi P + c^4\partial_\phi^2 P) \tag{7}$$

with $c = \cos \phi$ and $s = \sin \phi$.

This partial differential equation seems very hard to solve. However, if one is interested in the invariant measure $\mu(\phi)=\lim_{t\to\infty}\int_0^\infty P(I,\phi,t)\,dI$, one can integrate (7) by parts with respect to I and one finds the simpler equation

$$\partial_\phi \mu = D\partial_\phi[\cos^2\phi\,\partial_\phi(\cos^2\phi\,\mu)] \tag{8}$$

This equation is easily solved. Its normalizable solution is

$$\mu(\phi)=B(1+u^2)\int_u^{+\infty} du'\exp[-(1/D)(u'+u'^3/3)+(1/D)(u+u^3/3)] \tag{9}$$

where $u=\operatorname{tg}\phi$ and B is given by the condition

$$\int_0^{2\pi}\mu(\phi)\frac{d\phi}{2\pi}=1$$

Integrating (7) by parts with respect to ϕ leads to

$$\begin{aligned}\partial_t f = D\int [&4c^2s^2I^2\partial_I^2 + I(-4c^4+12s^2c^2+2c^2)\,\partial_I\\ &-2(c^4-3s^2c^2)]\,P(I,\phi,t)\,d\phi\end{aligned} \tag{10}$$

where we have set $f(I,t)=\int_0^{2\pi}P(I,\phi,t)\,d\phi$. Since $\log I$ is a homogeneous quantity of degree zero, the knowledge of $\mu(\phi)$ is sufficient to calculate the Lyapunov exponent. Indeed, multiplying (10) by $\log I$ and integrating by parts with respect to I, one derives the following equality:

$$\Lambda(D,1)=1/2\lim_{t\to\infty}\partial_t\langle\log I\rangle = D/2\int 2c^2(-1+2c^2)\,\mu(\phi)\,d\phi \tag{11}$$

Equations (9) and (11) completely determine the Lyapunov exponent. These results, which are exact, were obtained in a rather different way by Halperin[6] for the localization problem.

As a first example of the quasilinear approximation, let us show how it applies to this exactly solvable case. One way to introduce this method is to approximate the invariant measure by a constant: $\mu(\phi)=1/2\pi$. One can then easily compute the Lyapunov exponent by using (11). On this simple one-dimensional case, the quasilinear approximation is exactly in the same spirit as the random phase approximation used previously to study the localization problem on discrete lattices.[11]

In order to generalize this idea to more complicated case (in Section 3), let us introduce the quasilinear method in a different way. One first considers the Liouville equation satisfied by the density in phase space:

$$\partial_t\rho(I,\phi,t)+[H,\rho]=0 \tag{12}$$

Here the brackets denote the Poisson brackets:

$$[H, \rho] = \frac{\partial H}{\partial I}\frac{\partial \rho}{\partial \phi} - \frac{\partial H}{\partial \phi}\frac{\partial \rho}{\partial I}$$

The fact that the full measure in phase space is equal to $\langle \rho \rangle_V$, the mean of ρ with respect to the noise, is known as the Van Kampen lemma.[11]

Let us write $\rho = \langle \rho \rangle_{\phi, V} + \tilde{\rho}$, where $\tilde{\rho}$ is the fluctuating part of ρ and $\langle \rho \rangle_{\phi, V}$ denotes the mean of ρ with respect to the *angle and the noise*. Using the explicit form of the Hamiltonian, *and neglecting all terms of the form* $\tilde{\rho} V(t)$, (12) splits into two equations:

$$\partial_t f - \partial_I [I \langle \sin 2\phi \; V(t) \tilde{\rho} \rangle_{\phi, V}] = 0 \tag{13}$$

$$\partial_t \tilde{\rho} + \partial_\phi \tilde{\rho} - I \sin 2\phi \; V(t) \; \partial_I f = 0 \tag{13'}$$

Note that the first of these equations is exact. In this one-dimensional case with an uncorrelated noise, the quasilinear approximation consists essentially in the simple form of the second equation. Solving (13′) for ρ in term of f and substituting in (13), one obtains

$$\partial_t f = D/2 \; \partial_I I^2 \; \partial_I f \tag{14}$$

One then simply calculates the Lyapunov exponent of the system by multiplying (14) by $\log I$ and by integrating with respect to I.

Now some remarks are in order.

1. The Lyapunov exponent in the quasilinear approximation is simply

$$\Lambda_{\rm ql}(D, E) = \frac{\sqrt{E}}{4} \frac{D}{E^{3/2}} \tag{16}$$

2. The solution of (14) is a log-normal distribution:

$$P(I, \phi) = \frac{1}{(2\pi D t)^{1/2}} \exp\left[-\frac{(\log I - Dt)^2}{2Dt}\right] \tag{16}$$

3. When applied to the localization problem, our formulation and Melnikov's[13] are slightly different. Indeed, we compute here the Lyapunov exponent (which is obviously the same as in Melnikov's work), but rather than considering the measure characterizing the transfer matrix, we are concerned here with the measure in action–angle variables. That is why (16) differs from Melnikov's result.

4. Expanding the exact formula (9) for weak disorder, one finds

$$\mu(\phi)=\frac{1}{2\pi}\left[1-2\frac{D}{E^{3/2}}\frac{t}{(1+t^2)^2}\right] \tag{17}$$

Therefore the quasilinear approximation holds for $D \ll E^{3/2}$. In particular, it fails completely for $E=0$, that is, for degenerate harmonic oscillators. This is a general feature of this approximation. It holds only when the angle variable of the unperturbated system can rotate rapidly. It is this rotation which makes the measure over the angles isotropic.

One should also note that there is no correction at order D^2 to Λ_{ql}. Indeed, the correction at order D in (17) is odd and does not contribute to the Lyapunov exponent. However, it contributes to the density of states.

5. For $E=0$, the quasilinear approximation is false. Even the scaling in D is incorrect. However, one can calculate exactly the leading term of $\Lambda(D, 0)$ from (9) and (11). A careful computation of the limit $D/E \to \infty$ of Λ leads to

$$\Lambda(D, 0)=\frac{\sqrt{\pi}}{2}(12)^{1/3}\frac{1}{\Gamma(1/6)}D^{1/3} \tag{18}$$

This result is the same as the one obtained by Derrida and Gardner[14] for the Anderson localization at the band edge on the *discrete* one-dimensional lattice. This is not surprising, since at the band edge, the lattice discretization must have no effect.

2.2. One Degree of Freedom with a Correlated Gaussian Noise

In this part, the same system as in Section 2.1 is considered, but we suppose now that the perturbation is a stationary correlated noise with a correlation function $\langle V(t)\, V(t')\rangle = C(t-t')$. We are mainly interested in implementing the quasilinear method in this case and in deriving then a systematic way for computing the low-disorder expansion of the Lyapunov exponent when the correlation is exponential. One of our motivations is to study the behavior of this expansion in the two limits of large and small correlation time τ, in order to appreciate the validity of the quasilinear approximation.

The equations of motion are the same as in Section 2.1 [(13), (13′)]. One substitutes ρ from (13′) in (13) and, taking into account that at weak noise, the variations of f are slow, one gets the following simple diffusion equation:

$$\partial_t f = \tfrac{1}{2}\tilde{D}\, \partial_I I^2\, \partial_I f \tag{19}$$

with

$$\tilde{D} = \int_0^\infty C(t) \cos 2t \, dt \tag{20}$$

This result needs some comments:

1. In the particular case where the noise is exponentially correlated, i.e., $C(t) = \varepsilon^2 e^{-|t|/\tau}$, one obtains from (19) and (20)

$$\Lambda_{\text{ql}} = \frac{1}{4} \frac{\varepsilon^2 \tau}{1 + 4\tau^2} \tag{21}$$

One recovers that way the result of ref. 15. Note that the limit $\tau \to 0$ must be taken keeping $\varepsilon^2\tau$ constant: $\varepsilon^2\tau = D$.

2. As said above, the evolution of f is supposed to be slow, i.e., $\Lambda \ll 1/\tau$. From (21) a sufficient condition for this is $\varepsilon^2 \ll 1$. This condition, which is uniform in τ, will be discussed in more detail later. In physical units, this condition reads simply $\langle V^2 \rangle \ll E$.

3. In the limit $\tau \to \infty$ (i.e., slowly varying potential) it is possible to use a different method.[16] In this case one can eliminate the perturbation by performing successive canonical transformation, the number of which is fixed by the degree of differentiability of the perturbation. It is instructive to note that when applied to stationary processes, the simple result (21) indeed agrees with this adiabatic theory. In particular, if the correlation function is analytic everywhere, one gets from (19) and (21) a nonperturbative, at least exponentially small in $1/\tau$, Lyapunov exponent (we have in mind here a "deterministic" potential with the stationarity properties required to define a correlation function).

In the case of a Gaussian exponentially correlated noise one may analyze more quantitatively the validity of (21). In the following one constructs the expansion of Λ in powers of ε^2. In this case, the equations of motion are easily written in (I, ϕ) variables. Indeed, it is sufficient to add to Eq. (3) an additional equation describing the evolution of $y = V(t)$ considered as an auxiliary variable. To obtain an exponential correlation function for y it is sufficient to filter a Gaussian *white* noise $b(t)$. Therefore, we have

$$\begin{aligned} \dot{I} &= 2Iy \sin\phi \cos\phi \\ \dot{\phi} &= 1 + y \cos^2\phi \\ \dot{y} &= -\frac{y}{\tau} + \frac{b(t)}{\tau} \end{aligned} \tag{22}$$

where $b(t)$ is a Gaussian white noise with $\langle b(t)\, b(t')\rangle = 2D\delta(t-t')$. The system is now described by a joint measure $P(I, \phi, y, t)$. It satisfies a Fokker–Planck equation

$$\partial_t P + \partial_I(Iy \sin 2\phi P) + \partial_\phi(1 + y\cos^2\phi)P - \frac{1}{\tau}\partial_y(yP) = \frac{D}{\tau^2}\partial_y^2 P \quad (23)$$

Equation (23) is rather involved. However, the invariant measure $\mu(\phi, y) = \lim_{t\to\infty} \int P(I, y, \phi, t)\, dI$ obeys the simpler exact equation

$$\partial_\phi(1 + y\cos^2\phi)\mu - 1/\tau\, \partial_y(y\mu) = D/\tau^2\, \partial_y^2 \mu \quad (24)$$

Multiplying (23) by $1/2 \log I$, one obtains after integrating with respect to ϕ

$$\Lambda = 1/2\langle y \sin 2\phi\rangle \quad (25)$$

Unfortunately, it does not seem possible to solve (24) exactly.

However, (24) allows for a systematic perturbative scheme with respect to the strength of the disorder, to go beyond the quasilinear approximation. Indeed, if we set $\mu = F(y, \phi)\exp(-1/4\tau y^2/D)$, one finds

$$O[F] = \partial_\phi(1 + y\cos^2\phi)F \quad (26)$$

where the operator O is defined to be

$$O = \frac{D}{\tau^2}\partial_{y^2}^2 - \frac{y^2}{4D} + \frac{1}{2\tau} \quad (27)$$

Through the correspondence

$$h^2/2m = D/\tau^2, \qquad 1/2m\omega^2 = 1/4D$$

and apart from an additive constant, $-O$ is the Hamiltonian of the harmonic oscillator. Therefore, the eigenvalues of O are $o_n = -n/\tau$. The eigenvectors $\psi_n(y)$ are the same as for the harmonic oscillator. Thus, we expand $F(y, \phi)$:

$$F(y, \phi) = \sum a_n(\phi)\, \psi_n(y) \quad (28)$$

Substituting this expression in (24), one sees that the coefficients a_n are determined by the recurrence equations

$$\left(\frac{n}{\tau} + \partial_\phi\right) a_n + \frac{\varepsilon}{2}\,[(n+1)^{1/2}\,\partial_\phi(1 + \cos 2\phi)\, a_{n+1}$$
$$+ n^{1/2}\partial_\phi(1 + \cos 2\phi)\, a_{n-1}] = 0 \quad (29)$$

with $\varepsilon = (D/\tau)^{1/2}$. To solve perturbatively this infinite set of equations, one writes $a_n = \sum a_n^{(k)} \varepsilon^k$. Then one expands each $a_n^{(k)}$ in Fourier series,

$$a_n^{(k)} = \sum_m s_{n,m}^{(k)} \sin m\phi + c_{n,m}^{(k)} \cos m\phi \tag{30}$$

Thus, one is led to solve an infinite linear system:

$$\begin{aligned} &n/\tau s_{n,m}^{(k)} - m c_{n,m}^{(k)} \\ &\quad = m\varepsilon/2[(n+1)^{1/2} (c_{n+1,m}^{(k-1)} + 1/2 c_{n+1,m-2}^{(k-1)} + 1/2 c_{n+1,m+2}^{(k-1)}) \\ &\qquad + n^{1/2}(c_{n-1,m}^{(k-1)} + 1/2 c_{n-1,m-2}^{(k-1)} + 1/2 c_{n-1,m+2}^{(k-1)})] \\ &n/\tau c_{n,m}^{(k)} + m s_{n,m}^{(k)} \\ &\quad = m\varepsilon/2[(n+1)^{1/2} (s_{n+1,m}^{(k-1)} + 1/2 s_{n+1,m-2}^{(k-1)} + 1/2 s_{n+1,m+2}^{(k-1)}) \\ &\qquad + n^{1/2}(s_{n-1,m}^{(k-1)} + 1/2 s_{n-1,m-2}^{(k-1)} + 1/2 s_{n-1,m+2}^{(k-1)})] \end{aligned} \tag{31}$$

It is easy to see that, as expected, $s_{1,2}^{(2k)}$ is equal to zero for all k. The quasilinear approximation for Λ is determined by $s_{1,2}^{(1)}$. The first correction (of order ε^4) $\delta\Lambda^{(2)}$ is obtained by calculating $a_{1,2}^{(3)}$. Proceeding recursively, a straightforward but tedious calculation gives

$$\delta\Lambda^{(2)} = \left(\frac{D}{\tau}\right)^2 \tau^3 \frac{48\tau^4 + 22\tau^2 + 1}{4(\tau^2+1)(4\tau^2+1)^3} \tag{32}$$

Using the formal calculation program MACSYMA, we have been able to obtain the contribution of third order to Λ:

$$\delta\Lambda^{(3)} = \left(\frac{D}{\tau}\right)^3 \frac{5\tau^3 Q_7(\tau^2)}{64(\tau^2+1)^2 (4\tau^2+1)^5 (4\tau^2+9)(16\tau^2+1)} \tag{33}$$

$Q_7(\tau^2)$ is a polynomial of order seven in τ^2:

$$\begin{aligned} Q_7(\tau^2) = {} & 786{,}432\tau^{14} + 1{,}516{,}032\tau^{12} + 688{,}832\tau^{10} \\ & - 74{,}544\tau^8 - 97{,}596\tau^6 - 16{,}267\tau^4 - 1074\tau^2 - 27 \end{aligned}$$

In order to have more insight in the structure of the perturbative expansion of Λ, it is interesting to calculate at all orders in ε the dominant contribution in the limit $\tau \to \infty$. One considers the expansion of $\mu(\phi, y)$ in powers of $1/\tau$:

$$\mu(\phi, y) = \sum_{k=0}^{\infty} \mu_k(\phi, y) \left(\frac{1}{\tau}\right)^k \tag{34}$$

According to (24), $\mu_0(\phi, y)$ (which does not contribute to Λ) is determined by (here and in the following we rescale y as $y \to \varepsilon y$)

$$\partial_\phi(1+\varepsilon y\cos^2\phi)\mu = 0 \tag{35}$$

Therefore, $\mu_0(\phi, y) = h(y)/(1+y\cos^2\phi)$, where $h(y)$ is defined by

$$\int_0^{2\pi} \frac{d\phi}{1+\varepsilon y\cos^2\phi}\, h(y) = \frac{1}{(2\pi)^{1/2}}\, e^{-y^2/2} \tag{36}$$

namely

$$h(y) = \frac{1}{(2\pi)^{3/2}}\, e^{-y^2/2}(1+\varepsilon y)^{1/2}$$

One gets $\mu_1(\phi, y)$ by solving

$$\partial_\phi(1+\varepsilon y\cos^2\phi)\,\mu_1 = 1/\tau(\partial_y^2 + \partial_y y)\,\mu_0 \tag{37}$$

Inverting this equation and using (24), one may write the correction of order $1/\tau$ to the Lyapunov exponent as

$$\begin{aligned}\delta\Lambda_1 = \frac{1}{\tau}\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{3/2}}\,dy \int_0^{2\pi} d\phi\, \frac{\sin 2\phi\; y}{1+\varepsilon y\cos^2\phi} \\ \times \int^{\phi} d\phi'\,(\partial_y^2 + \partial_y y)\, \frac{e^{-y^2/2}(1+\varepsilon y)^{1/2}}{1+\varepsilon y\cos^2\phi'}\end{aligned} \tag{38}$$

Integrating by parts with respect to ϕ and y, one obtains

$$\delta\Lambda_1 = -\frac{1}{\tau(2\pi)^{3/2}}\int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dy\, \frac{\varepsilon^2\cos^4\phi + y\varepsilon\Delta\cos^2\phi}{\Delta^2}\, e^{-y^2/2}\, \frac{(1+\varepsilon y)^{1/2}}{\Delta} \tag{39}$$

where $\Delta = 1+\varepsilon y\cos^2\phi$. Now, a trick allows us to simplify this expression. Indeed, one notes that (39) can be rewritten as

$$\delta\Lambda^{(1)} = -\frac{1}{\tau}\int_{-\infty}^{+\infty} \frac{e^{-y^2/2}}{(2\pi)^{3/2}}\,dy \int_0^{2\pi} d\phi\,(1+\varepsilon y)^{1/2}\left(-\varepsilon\partial_\varepsilon + \frac{\varepsilon^2}{2y^2}\partial_\varepsilon^2\right)\frac{1}{\Delta} \tag{40}$$

Integration over ϕ can be explicitly performed, and after some manipulations one obtains the rather simple formula

$$\delta\Lambda_1 = -\frac{1}{8}\int_{-\infty}^{+\infty} dy\, \frac{\varepsilon y}{1+\varepsilon y}\, \frac{e^{-y^2/2}}{(2\pi)^{1/2}} \tag{41}$$

This result must be considered as formal. Indeed, due to the singularity of the integrand, the integral does not exist. However, (41) allows us to compute the asymptotic perturbative expansion in ε of $\delta\Lambda_1$:

$$\delta\Lambda_1 = \frac{1}{8}\sum_k 2^k \frac{\Gamma((2k+1)/2)}{\sqrt{\pi}}\,\varepsilon^{2k} \tag{42}$$

The singularity in (41) is a consequence of large fluctuations ($|y| \gg 1/\varepsilon$), which are implicitly excluded in the definition of μ_0. This singularity must be canceled by a nonpertubative contribution to Λ.

Taking into account all the results of this part, one can see that the perturbative expansion of Λ is an asymptotic series in ε^2, each term of which is bounded uniformly in τ. Therefore one concludes that the condition $\varepsilon \ll 1$ is indeed sufficient for all τ for the quasilinear approximation to hold.

3. GENERAL FORMULATION FOR *n* DEGREES OF FREEDOM

3.1. The Quasilinear Fokker–Planck Equation

In the following, we formulate the quasilinear method for Hamiltonian systems with an arbitrary number of degrees of freedom driven by a correlated stationary Gaussian noise. As we will show, this method allows one to obtain a low-disorder equation for the distribution of the actions of the system (when it is linear). In some cases this will also provide a low-disorder invariant measure. Once the invariant measure is known, it is not difficult to obtain the quasilinear largest Lyapunov exponent of the system.

Let us consider a system whose Hamiltonian reads

$$H(I, \phi) = H_0(I) + V(I, \phi, t)$$

$I = (I_i)$, the actions $\phi = (\phi_i)$, the phases $i = 1,..., n$. The stochastic perturbation $V(I, \phi)$ may be expanded over its Fourier components:

$$V(I, \phi, t) = \sum_{m,\alpha} V_m^\alpha(I)\, e^{im\phi} b_\alpha(t) \tag{43}$$

the $b_\alpha(t)$ ($\alpha = 1,..., n$) are n Gaussian noises with zero means and correlation functions

$$\langle b_\alpha(t)\, b_\beta(t')\rangle = \delta_{\alpha,\beta} C^\alpha(t-t') \tag{44}$$

$m = (m_i)$, $i = 1,...,$ is a n-integer-component vector. The Fourier transform of $C^\alpha(t)$ will be denoted $C^\alpha(\omega)$. The Liouville equation satisfied by the measure $\rho(I, \phi, t)$ for a given realization of V is

$$\partial_t \rho(I, \phi, t) + [H, \rho] = 0 \tag{45}$$

As in Section 2, one splits ρ into two parts:

$$\rho(I, \phi, t) = \langle \rho \rangle_{\phi, V} + \tilde{\rho} \tag{46}$$

and the exact equation satisfied by $f(\{I_j\}) = \langle \rho \rangle_{\phi, V}$ is

$$\partial_t f - i \sum_{j,m,\alpha} \partial_{I_j} \langle e^{im\phi} V^\alpha_m(I)\, m_j b_\alpha(t) \rho \rangle = 0 \tag{47}$$

ρ is given by the approximate equation obtained by neglecting all terms of the form $\rho b(t)$. One obtains

$$\partial_t \tilde{\rho} + \sum_j \Omega_j \partial_{\phi_j} \tilde{\rho} - i \sum_{j,m,\alpha} m_j V^\alpha_m(I)\, e^{im\phi} b_\alpha(t)\, \partial_{I_j} f = 0 \tag{48}$$

The solution of this equation is

$$\tilde{\rho} = i \sum_{m,j,\alpha} e^{im(\phi - \Omega t)} \int_0^t m_j V^\alpha_m(I)\, b_\alpha(t')\, \partial_{I_j} f(t')\, e^{im\Omega t'}\, dt' \tag{49}$$

Substituting this expression in (47) and taking the limit $t \to \infty$, one obtains under the assumption of a slow temporal variation of f

$$\partial_t f = \frac{1}{2} \sum_{i,j} \partial_{I_i} D_{ij} \partial_{I_j} f \tag{50}$$

with

$$D_{ij} = \sum_{\alpha,m} m_i m_j\, |V^\alpha_m|^2\, C^\alpha(m, \Omega) \tag{51}$$

This equation determines the quasilinear approximation for the action distribution in the limit $t \to \infty$. It holds for any Hamiltonian system provided: (1) it is nondegenerate, that is, all the Ω_i are all different and not equal to zero. Indeed, it is only under this assumption that the averaging over the phases is legitimate. In particular, the validity of this approach is questionable in the thermodynamic limit. This point will be considered further in Section 5.3. (2) The noise V is weak. Furthermore, it is assumed that the conclusion of Section 2.2 holds also in this more general case: it is reasonable to suppose that the validity of the quasilinear approximation is uniform in the correlation times as in the one-dimensional case. However, we have not proved this point.

3.2. Linear Hamiltonian Systems

Now, we specialize to the more specific case of linear Hamiltonian systems, defined by quadratic Hamiltonians. In this case, the Hamiltonian written in action–angle variables is

$$H=\sum_i \Omega_i I_i+\sum_{i,j,\alpha}(I_i I_j)^{1/2}\cos\phi_i\cos\phi_j\, A^\alpha_{ij} b^\alpha(t) \tag{52}$$

A is an $n\times n$ real matrix which depends on the system under consideration. In this case

$$V^\alpha_m(I)=\frac{1}{2}\sum_i I_i A^\alpha_{ii} \qquad \text{for}\quad m_i=0 \text{ for all } i$$

$$V^\alpha_m(I)=\frac{1}{4} I_i A^\alpha_{ii} \qquad \text{for}\quad m_i=+2,\ -2 \text{ and } m_j=0 \text{ for } i\neq j$$

$$V^\alpha_m(I)=\frac{1}{2}(I_i I_j)^{1/2} A^\alpha_{ij} \qquad \text{for}\quad m_i=1,\ -1,\quad m_j=+1,\ -1$$

$$\text{and } m_k=0\quad \text{for } k\neq i \text{ and } k\neq j$$

Therefore, according to (50), the diffusion matrix is defined by

$$D_{ii}=\frac{1}{2}\left\{I_i^2\,|A^\alpha_{ii}|^2\,C^\alpha(2\Omega_i)+\sum_{i\neq j} I_i I_j\,|A^\alpha_{ij}|^2\,[C^\alpha(\Omega_i+\Omega_j)+C^\alpha(\Omega_i-\Omega_j)]\right\} \tag{53}$$

$$D_{ij}=\frac{1}{4} I_i I_j\,|A^\alpha_{ij}|^2\,[C^\alpha(\Omega_i+\Omega_j)-C^\alpha(\Omega_i-\Omega_j)] \tag{54}$$

For the sake of brevity we set in the following

$$\begin{aligned}\alpha_i&=1/4\,|A^\alpha_{ii}|^2\,C^\alpha(2\Omega_i)\\ \beta_{ij}&=1/4\,|A^\alpha_{ij}|^2\,[C^\alpha(\Omega_i+\Omega_j)+C^\alpha(\Omega_i-\Omega_j)]\\ \gamma_{ij}&=1/4\,|A^\alpha_{ij}|^2\,[C^\alpha(\Omega_i+\Omega_j)-C^\alpha(\Omega_i-\Omega_j)]\end{aligned} \tag{55}$$

where the summation over α is implicit.

With these notations (50) is written

$$\partial_t f=\left(\sum \alpha_i\,\partial_{I_i} I_i^2\,\partial_{I_i}+\beta_{ij}\,\partial_{I_i} I_i I_j\,\partial_{I_i}+\sum \gamma_{ij}\,\partial_{I_i} I_i I_j\,\partial_{I_j}\right) f \tag{56}$$

At large time $f(\{I_i\},t)$ does not have a limit. However, one can define the projection μ of f on an $(n-1)$-sphere (more precisely, in a sector of this sphere where all the I_i are positive),

$$\mu(\{I_i\}) = \int_0^\infty r^{n-1} f(\{rI_i\})\, dr \tag{57}$$

with $I_i = I_i/r$ and $r^2 = \sum I_i^2$. From general theorems,[8] μ is ensured to have a limiting value, namely the invariant measure. Now, the largest Lyapunov exponent is given by

$$\Lambda_{\max} = 1/2 \lim_{t\to\infty} \partial_t \langle \log \|I\| \rangle \tag{58}$$

where $\|\cdot\|$ may be any norm equivalent to the Euclidean norm. Note, in particular, that $\partial_t \langle \log I_i \rangle$ *is not* the largest Lyapunov exponent. Obviously, the homogeneity of the integrands allows one to compute $\Lambda_{\max}$ once μ is known. As will be seen later, $\Lambda_{\max}$ is easily computable in the thermodynamic limit in term of the matrix $(\partial^2/\partial_{I_i}\partial_{I_j})\, D_{ij}$.

Let us show how to compute the quasilinear evolution of the whole set of momenta $\langle I_i \rangle$ $(i = 1,..., n)$. Using (50), one finds in the limit $t \to \infty$ and after integrating by parts

$$\langle \dot{I}_k \rangle = \sum_j \left\langle \frac{\partial D_{kj}}{\partial I_j} \right\rangle \tag{59}$$

Therefore, the behavior of these moments at large t is completely determined by the largest eigenvalue of the matrix Δ:

$$\Delta_{kk} = 2\alpha_k + \sum_{j\neq k} \gamma_{kj} \tag{60}$$

$$\Delta_{kj} = \Delta_{jk} = \beta_{jk} \tag{61}$$

So the largest eigenvalue of Δ is nothing else than the so-called "generalized Lyapunov exponent of order two," which we denote by Λ_2:

$$\Lambda_2 = \lim_{T\to\infty} \frac{1}{2T} \log\langle \|I\| \rangle \tag{62}$$

Let us prove that the quasilinear diffusion equation provides a positive Λ_2. The largest eigenvalue of Δ, $\lambda_{\max}$, is real and strictly positive. Indeed, Δ is symmetric. So all its eigenvalues are real. The largest eigenvalue of Δ is $\sup\langle X|\Delta|X\rangle/\langle X|X\rangle$. Let X_0 be the vector with all its coordinates equal to 1. It is clear that $\Delta X_0 > 0$. Therefore, $\lambda_{\max}$ is positive. Now, it is possible to choose $\eta > 0$ so that $\Delta = \eta I + \Delta > 0$ and $\eta + \lambda_{\max}$ is the largest eigenvalue of Δ. Hence, the Perron theorem asserts that its eigenspace is of dimension one and contains a vector with all its coordinates positive. Obviously this is also true for Δ.

To conclude this section, let us stress that the other eigenvalues of Δ have no physical meaning, in particular, they can be all negative.

3.3. Lowest Order for the Sum of the Positive Lyapunov Exponents

The sum of the n positive Lyapunov exponents for a general Hamiltonian system with n degrees of freedom driven by a stationary correlated noise is easily computable at least in the weak-noise limit. This quantity is sometimes called the Kolmogorov entropy of the system and will be denoted by Σ. The method we use was presented elsewhere[9] for the case of an uncorrelated noise. For the sake of completeness we sketch here the essential features of the method.

The equations of motion are

$$\frac{\partial H}{\partial Q_i} = -\dot{P}_i, \qquad \frac{\partial H}{\partial P_i} = \dot{Q}_i \tag{63}$$

Equivalently, we introduce the evolution operator U of the system. It satisfies

$$\dot{U} = \begin{pmatrix} 0 & I \\ -H_0 + V & 0 \end{pmatrix} U \tag{64}$$

It is easy to see that one can find a basis where the equation of motion of U is

$$\dot{U} = \begin{pmatrix} -i\Omega & 0 \\ 0 & i\Omega \end{pmatrix} U + \frac{i}{2} \begin{pmatrix} W & W \\ -W & -W \end{pmatrix} U \tag{65}$$

where we have set $\Omega = H_0^{1/2}$, $W = \Omega^{-1/2} V \Omega^{-1/2}$.

This basis can also be chosen so that H is diagonal. The first term of (65) may be eliminated by working in a rotating basis (with "frequency" Ω). Writing in this basis the evolution operator

$$U = \begin{pmatrix} X^- & Y^- \\ X^+ & Y^+ \end{pmatrix}$$

where X^ε and Y^ε, $\varepsilon = +1$ or -1, are $n \times n$ matrices, one obtains

$$\dot{X}^- = \frac{i}{2} W^{-+} X^- + \frac{i}{2} W^{--} X^+, \qquad \dot{X}^+ = -\frac{i}{2} W^{+-} X^+ - \frac{i}{2} W^{++} X^- \tag{66}$$

Here

$$W^{\varepsilon\varepsilon'} = e^{i\varepsilon\Omega t} W e^{i\varepsilon'\Omega t}$$

In order to compute the Kolmogorov entropy we argue that

$$\Sigma = 1/2\langle \partial_t \log \operatorname{Det} X^+ X^- \rangle \tag{67}$$

Using the Wronski identity and Eq. (66), one obtains

$$\Sigma = -1/2 \operatorname{Re}\langle i \operatorname{Tr} W^{++} Z\rangle \tag{68}$$

with $Z = X^-(X^+)^{-1}$.

It is straightforward to see that Z satisfies

$$\dot{Z} = \frac{i}{2} W^{-+} Z + \frac{i}{2} Z W^{+-} + \frac{i}{2} Z W^{++} Z + \frac{i}{2} W^{--} \tag{69}$$

The contribution of lowest order in D to Σ is easy to obtain. Formally integrating (69), one can write

$$\langle \operatorname{Tr} W^{++} Z\rangle$$
$$= \left\langle \operatorname{Tr} W^{++} \left[Z(0) + \frac{i}{2} \int_0^t (W^{-+} Z + W^{--} + Z W^{+-} + Z W^{++} Z)\, dt \right] \right\rangle \tag{70}$$

To the lowest order only one term contributes, namely the term with no Z. All the other terms are at least of order D^2. Expressing W in term of the structure matrix A^α and coming back to the notations of Section 3.2, one finds after some algebra

$$\Sigma = \frac{1}{8} \sum_{i,j,\alpha} C^\alpha(\Omega_{ii} + \Omega_{jj})\, |A^\alpha_{ij}|^2 + O(D^2) \tag{71}$$

With the notations of (55) this reads

$$\Sigma = \frac{1}{2} \sum_i \alpha_i + \frac{1}{4} \sum_{i,j} (\beta_{ij} + \gamma_{ij}) = \frac{1}{4} \sum_{i,j} \Delta_{ij} \tag{71'}$$

In particular, in the one-dimensional correlated case one recovers as, one should, the result of (21):

$$\Sigma_{\mathrm{ql}} = \Lambda_{\mathrm{ql}} = \frac{D}{4} \frac{1}{1 + 4\tau^2}$$

Note that if there is no correlation, the perturbative expansion of Σ can be performed at higher order systematically as explained in ref. 9. In the correlated case this is also possible, but the calculations are more involved. It is important to stress that (71) does not require any hypothesis on the level spacing.

4. HAMILTONIAN WITH A FINITE NUMBER OF DEGREES OF FREEDOM

4.1. Two Degrees of Freedom

The two-channel localization problem was addressed previously by Dorokhov[10] in the limit of vanishing coupling between the channels and low disorder. We show in the following that, in fact, it may be completely and easily solved at all coupling and low disorder, even in the correlated case, using the quasilinear approximation.

Making the change of variable $I_1 = r\cos\theta$ and $I_2 = r\sin\theta$ with $\theta\in[0,\pi/2]$, we consider the projection of $f(r,\theta,t)$ on the unit sphere, namely $\mu(\theta,t) = \int rf(r,\theta,t)\,dt$. The equation for μ is easy to obtain from (56). Integrating by parts with respect to r, it is straightforward to obtain

$$\partial_t\mu = \partial_\theta\{2[(\alpha_1-\gamma)\,sc^3 - (\alpha_2-\gamma)\,s^3c]\,\mu + [(\alpha_1+\alpha_2-2\gamma)\,s^2c^2 + \beta sc]\,\partial_\theta\mu\} \tag{72}$$

with $\beta_{12}=\beta_{21}=\beta$, $\gamma_{12}=\gamma_{21}=\gamma$, $s=\sin\theta$, and $c=\cos\theta$. The invariant measure is computed by setting the right-hand side of (72) to zero. Hence

$$\mu(\theta) = (1+u^2)\frac{K}{\beta(1+u^2)+(\alpha_1+\alpha_2-2\gamma)u}\,e^{\phi(u)} \tag{73}$$

with

$$\phi(u) = \int_0^u \frac{\alpha_2-\alpha_1}{\beta(1+u^2)+(\alpha_1+\alpha_2-2\gamma)u}\,du$$

where $u=\operatorname{tg}\theta$ and K is some normalization constant [note that $\phi(u)$ is computable in a closed form]. In the following we will denote by $h(u)$ the function $\mu(\theta)/(1+u^2)$.

As said earlier, the knowledge of $\mu(\theta)$ is sufficient for calculating the largest Lyapunov exponent of the system. Indeed, multiplying the equation satisfied by f by $\log(I_1^2+I_2^2)$ and integrating (by parts) with respect to I_1 and I_2, one obtains

$$\Lambda = \frac{1}{2}\left\{\int_0^\infty \frac{h(u)}{(1+u^2)^2}\left[\alpha_1(1+3u^2)+\alpha_2(3u^2+u^4)\right.\right.$$
$$\left.\left. +\gamma(1-u^2)^2+2\beta u(1+u^2)\right] du\right\} \tag{74}$$

(73), (74), and (71) solve completely the problem in the quasilinear approximation with arbitrary coupling between the channels except in the strictly degenerate case where the two frequencies are equal.

Two limits are of some interest.

1. Let us consider two oscillators coupled only through a noise (each of them being also perturbed by a noise). The relative excitation level of these oscillators can be characterized by the ratio $\mu(0)/\mu(\pi/2)$. For small coupling ($\beta \to 0$) and $\gamma = 0$ (no correlation) one finds

$$\frac{\mu(0)}{\mu(\pi/2)} = \left(\frac{\beta}{\alpha_1+\alpha_2}\right)^{2(\alpha_2-\alpha_1)/(\alpha_1+\alpha_2)}$$

Another question of interest is the repulsion of the two positive Lyapunov exponents as one switches on the coupling between the two oscillators. One considers the nearly degenerate case $\alpha_1 \approx \alpha_2 = \alpha$ (the strictly degenerate case cannot be handle under the phase averaging approximation). Using (74) and (71), one finds for the two Lyapunov exponents

$$\begin{aligned} \Lambda_{\max}(\beta) &= \frac{\alpha}{2}\left[1+\frac{1}{\log(2\alpha/\beta)}+O(\beta)\right], \\ \Lambda_{\min}(\beta) &= \frac{\alpha}{2}\left[1-\frac{1}{\log(2\alpha/\beta)}+O(\beta)\right] \end{aligned} \tag{75}$$

This result is valid only if $\alpha_1-\alpha_2 \ll \alpha/\log(2\alpha/\beta)$. However, one can hope to find the same behavior in the exactly degenerate case.

Hence, due to the effect of the coupling one gets a strong and non-analytic (in β) repulsion of the two Lyapunov exponents. This property can be traced back to the more general behavior expected for the Lyapunov spectrum in disordered systems.[17] It is amazing that this repulsion is stronger than the repulsion of energy levels in systems with small coupling.

2. The two-channel Anderson localization problem corresponds to the following choice of coefficients:

$$\alpha_1 = \frac{D}{4(E-\lambda)^{1/2}}, \qquad \alpha_2 = \frac{D}{4(E+\lambda)^{1/2}}, \qquad \beta = \frac{D}{2(E^2-\lambda^2)^{1/4}} \tag{76}$$

where E is the energy of the system and λ is the coupling between the channels. In the limit where the energy is large with respect to the coupling, $\alpha_1 \approx \alpha_2 \approx \beta/2$, the two positive Lyapunov exponents are

$$\Lambda_{\max} = \frac{D}{8E}\left(1 + \frac{3\sqrt{3}}{\pi}\right), \qquad \Lambda_{\min} = 3\frac{D}{8E}\left(1 - \frac{\sqrt{3}}{\pi}\right) \tag{77}$$

According to (71), one must have at low disorder $\Lambda_{\max} + \Lambda_{\min} = 2\Lambda_0(D, E)$, where $\Lambda_0(D, E)$ is given by (15). This result is in agreement with formula (51) of Dorokhov[1] (put $\beta = \gamma$ in Dorovkov's result).

It is interesting to note that the measure does not vanish on the boundary $I_1 = 0$, $I_2 \neq 0$ or $I_2 = 0$, $I_1 \neq 0$. In particular, the quantity $\langle I_1/I_2 \rangle$ is infinite in the quasilinear approximation.

4.2. Finite Number of Degrees of Freedom

In this section we make some comments on the general case with a finite number of degree of freedom.

For $n = 2$ (Section 2.3) one is led to a one-dimensional equation for the invariant measure. For $n > 2$, the invariant measure at the quasilinear level is defined on an $(n-1)$-sphere embedded in the I_i space (more precisely, in the sector with all the I_i positive). Even for $n = 3$ we did not succeed in obtaining μ in the general case. In our opinion this is not fortuitous, as can be argued as follows.

It is possible to deal with dynamical systems driven by white noise through a path integral approach. The problem is reduced to the evaluation of a partition function for an effective Hamiltonian. In the case of linear systems this Hamiltonian is quartic. For example, the Hamiltonian after a quasilinear phase averaging has the structure

$$H(\{I_i\}, \{J_i\}) = \sum_{i,j} I_i I_j J_i^2 \beta_{ij}$$

(J_i are the conjugate variables of the I_i). This Hamiltonian is integrable for $n = 2$ due to a simple scale invariance. For $n > 2$ it seems to be not integrable except for some isolated values of β_{ij}.[18] It is reasonable to think that the integrability of H is required for an analytical computation of the Lyapunov exponent to be possible.

However, in the particular case where there is no correlation and where $\beta_{ij} = \alpha_i$ $(=\alpha)$ for all i, j, Eq. (56) may be explicitly solved. One finds

$$f(I_i, t) = \frac{K}{\sqrt{t}} e^{-[n\alpha t + \log \Sigma]^2/4\alpha t} \tag{78}$$

with $\Sigma = \sum I_i$, and K is a normalization constant. The corresponding invariant measure is rather simple:

$$\mu \propto (1/\Sigma)^n$$

The Lyapunov exponent is

$$\Lambda_{\max} = \frac{1}{2} n\alpha$$

Restricted to this integrable case, one may calculate the various moments of the I_i. An interesting quantity is the generalized Lyapunov of order p defined by

$$\Lambda_{2p} = \lim_{t \to \infty} \partial_t \log\langle \Sigma^p \rangle / 2p$$

One finds

$$\frac{\Lambda_{2p}}{\Lambda_{\max}} = 1 + \frac{p}{n} \tag{79}$$

Hence, at the integrable point there is no dispersion for small p. In fact, as shown in the next part, this property holds in the more general cases in the large-n limit.

5. LARGE-n LIMIT

One considers in what follows the general n-degree-of-freedom problem. For the sake of simplicity the noise is supposed uncorrelated. As argued in the preceding part, this problem does not seem to be solvable. However, it is possible to obtain results in the large-n limit.

Let us rewrite the equation satisfied by the measure in the more compact form

$$\partial_t f = \sum a_{ij} \partial_{I_i} I_i I_j \partial_{I_i} P \tag{80}$$

We set $\Sigma = \sum \lambda_i I_i$ and $X_k = \langle I_k / \Sigma \rangle$; the coefficients λ_i are real, *strictly positive*, and such that $\sum \lambda_i^2 = 1$. They will be completely determined later.

As the λ_i are strictly positive, Σ is a norm of (I_i). Thus $1/2\, \partial_t \log \Sigma$ is equal to the largest Lyapunov exponent, in the limit $t \to \infty$. Therefore, using (80), one has

$$2\Lambda_{\max} = \sum_{i,j} \lambda_i a_{ij} X_j + \sum_i \lambda_i a_{ii} X_i - \sum_{i,j} \lambda_i^2 a_{ij} \left\langle \frac{I_i I_j}{\Sigma^2} \right\rangle \tag{81}$$

Now, Σ is of order $\sqrt{n}$, a_{ij} is of order 1, and X_i is of order $1/\sqrt{n}$. Assuming that for all i and j, $\langle I_i I_j / \Sigma^2 \rangle$ are of the same order, they are necessarily of order $1/n$ and the last term of (81) is of order 1 while the first is of order n. Thus, only the first term of (81) is dominant in the limit $n \to \infty$. To complete the specification of the coefficients λ_i we choose the vector (λ) to be an eigenvector of the matrix $\Delta = a_{ij} + \delta_{ij} a_{ii}$ with eigenvalue equal to the largest one. (This is indeed possible, as the Perron–Frobenius theorem asserts that the largest eigenvalue ν of the strictly positive matrix a_{ij} is real and strictly positive, and that its eigenspace is of dimension 1 and is spanned by a vector with all its components strictly positive. Moreover, as Δ is symmetric, any other strictly positive eigenvector is in this eigenspace.) Thus, we make the choice

$$\sum \lambda_i (a_{ij} + \delta_{ij} a_{ii}) = \nu \lambda_j, \qquad \sum \lambda_i^2 = 1 \tag{82}$$

With this set of λ_i one finds

$$\Lambda_{\max} = 1/2[\nu + O(1/n)] \tag{83}$$

As shown in Section 3, the largest eigenvalue of Δ is Λ_2. Therefore

$$\lim_{n \to \infty} \frac{\Lambda_{\max}}{\Lambda_2} = 1 \tag{84}$$

The naive ordering used in the discussion of (81) needs some sufficient coupling. In order to have more insight into this last point, let us prove it in a more rigorous way. The idea relies on the fact that Λ_p is an increasing and a convex function of p and on computing Λ_2 and Λ_4. The Λ_2 is given by (62) while Λ_4 is given by the eigenvalue problem

$$\Delta_{il} \xi_{jl} + \Delta_{jl} \xi_{il} + 2\delta_{ij} a_{il} \xi_{il} = 4\Lambda_4 \xi_{ij} \tag{85}$$

where ξ_{ij} is a symmetric tensor. Without the third term of the left-hand side of (85) one would get $\Lambda_2 = \Lambda_4$. One can now compute perturbatively the effect of the third term of (85) on Λ_4. One finds

$$\frac{\Lambda_4 - \Lambda_2}{\Lambda_2} = \sum_i \lambda_i^4 + O\left(\frac{1}{n^2}\right) \tag{86}$$

where λ_i is given by (82). The coupling will be sufficient for the matrix Δ such that $\sum \lambda_i^4$ is of order $1/n$. Equation (86) is a generalization of (79) for $p=2$. Thus, the remarkable property (79) of the integrable case of Section 4.2 does not seem exceptional. One expects a behavior similar to (79) for all p.

5.1. A Comment on the Lyapunov Spectrum

Taking into account the results of Sections 5.1 and 3.3, one can evaluate in the thermodynamic limit the ratio $I = n\lambda_{\max}/\Sigma$, which we call the index of the Lyapunov spectrum:

$$I = 2N \frac{v(\Delta)}{\sum_{i,j} a_{ij}} \tag{87}$$

I is obviously larger than two $[v(\Delta) = \sup\langle X|\Delta|X\rangle / \langle X|X\rangle$ and the result follows from the special choice $X_i = 1$ for all $i]$. This result excludes the occurrence of a concave spectrum of Lyapunov exponents at weak noise. This is in agreement with previous numerical results.[4] On the other hand, a rigorously linear spectrum can occur only if additional symmetry properties of Δ are satisfied.[5]

5.2. Application to the n-Channel Localization Problem

For the sake of completeness, we recall here the main results we obtained in a previous work[9] for the n-channel localization problem in the large-n limit. More details can be found in ref. 9. In that paper, the exact equation satisfied by the invariant measure was used to show that in the large-n limit and at weak noise, the largest Lyapunov exponent and the generalized Lyapunov exponent Λ_2 were equal. This result can be recovered by using the quasilinear method. However, the validity of this approach is questionable in the limit $n \to \infty$. It was argued that this result holds up to correction of order $D/\Delta E$, where ΔE is the width of the energy spectrum of the unperturbed system. It is important to note that the much more stronger condition $D/\delta E$, where δE is the mean spacing between energy levels $(\delta E \approx \Delta E/n)$, is not required. This ensures that the quasilinear result of the absence of dispersion is indeed valid in the large-n limit.

The sense of the quasilinear method in the thermodynamic limit is not obvious. Indeed, in this limit the energy level spacing vanishes and for a general system this would impose taking $D \to 0$. However, one can convince oneself that if the system possesses some particular properties, the quasilinear theory subsists for $D \gg \delta E$. An example of a physical system which has such good properties is the n-channel random Schrödinger equation. The proof given for this case in ref. 9 suggests that the sufficiency of the two following properties may be conjectured:

1. An invariance property of the Hamiltonian and of the noise structure (in the Schrödinger case: the invariance by translation).

2. The fact that the random perturbation is a sum of n projectors [in (49), A^α is the matrix of a projector].

A result similar to (87) has been obtained in ref. 9: the index of the spectrum defined by $I = n\Lambda_{\max}/\Sigma$ is given in term of the (transverse) density of states of the unperturbed system $\rho(\varepsilon)$ by the formula

$$I = 2\left[\int \frac{\rho(\varepsilon)}{\varepsilon}\, d\varepsilon \int \rho(\varepsilon)\, d\varepsilon\right] \Big/ \left(\int \frac{\rho(\varepsilon)}{\sqrt{\varepsilon}}\, d\varepsilon\right)^2 \tag{88}$$

It follows that for weak disorder I is always greater than 2. At high energy compared to the transverse coupling, $I = 2$, but at low energy one has in general $I \neq 2$. This analytical result supports the idea previously suggested on the basis of numerical simulations[4] that the Lyapunov spectrum in the thermodynamic limit is always convex at small noise (a localization being related to a threshold for the concavity to occur). In the case of the one-dimensional Anderson localization, $\Lambda_{\max} = 1/2\Lambda_2$. It is important to stress that it is precisely the absence of dispersion in the large-n limit which makes $I = 2$ at high energy.

6. CONCLUSION

The aim of this work was to discuss, on some examples, the use of diffusion equations in the study of Hamiltonian systems driven by noise. As far as one is concerned with the "direct" problem of parametric oscillators and thus interested in the largest Lyapunov exponent in the usually studied (and in general physically relevant) limit of weak noise, these methods appear to be rather powerful. In one dimension, the white noise problem is exactly solvable, while in the correlated noise case, these techniques provide an asymptotic expansion rather well behaved and sufficient for practical use. In dimension n greater than one, the evaluation of the largest Lyapunov exponent is also possible either through a calculation of an exact (small-noise) invariant measure (as in the two-dimensional case or in some exceptional case for $n > 2$) or by a large-n estimation. For $n = \infty$, the difficulty which arises from the degeneracy can be overcome by a very similar method[9] which does not require a nondegeneracy hypothesis.

For the localization problem, in which all the Lyapunov exponents are relevant, the method of this paper can only give more insight into the convexity of the Lyapunov spectrum, but it does not solve the problem completely (except for two coupled channels). Though Dorokhov[1,10] applied similar methods for the whole transfer matrix, his analysis is restricted to small coupling.

We have emphasized in this work the quasilinear formalism because the diffusion equations obtained in this framework are algebraically simple and nevertheless they contain striking features of the problem: non-solvability except in particular cases, remarkable properties in the thermodynamical limit.

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